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Approximation by smooth curves near the tangent cone

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Abstract

We show that through a point of an affine variety there always exists a smooth plane curve inside the ambient affine space, which has the multiplicity of intersection with the variety at least 3. This result has an application to the study of affine schemes.

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This note appeared as a result of studying certain affine schemes. Namely, I.R. Shafarevich in [3] studied the question: which irreducible components of the reduced scheme C_n^{red} of associative commutative multiplications on a n -dimensional vector space consist of those multiplications that represent nilpotent degree 3 algebras? While this problem was partially solved in [3] and [1], the methods used there do not suffice to give a complete answer to the question. The basic idea, due to Shafarevich, was to compare the tangent spaces to the non-reduced scheme C_n and the scheme A_n of nilpotent degree 3 commutative multiplications on a n -dimensional vector space. In certain cases, the difference of the tangent vectors was explained by the presence of nilpotents in the structure sheaf of the scheme C_n which can be eliminated by embedding C_n into the scheme of multiplications on a $(n + 1)$ -dimensional vector space representing associative algebras with a unit. Another method, used in [1], is to apply “obstructions to deformations” of algebras. However, this approach required the variety C_n^{red} to be smooth at a point of A_n^{red} (a priori, we do not know if this is true) in order to deduce a definite answer to Shafarevich’s

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question. The result of this paper shows that we can still use the second order obstructions to deformations of algebras, even if a whole irreducible component of A_n^{red} is in the singular locus of C_n^{red} .

Our plan is to introduce the multiplicity of intersection of a smooth curve with a variety at a point in Section 1, and then show that an affine variety can be nicely “approximated” by smooth plane curves inside the ambient affine space. Section 2 discusses an application of Theorem 1.3 to the studying of affine schemes of algebras.

1. Plane curves approximating varieties near the tangent cone

In this section, we generalize the notion of multiplicity of intersection used in [2, Chapter 3, §4, Definition 1]. Proposition 3 in [2, Chapter 9, §6], has a criterion which says that a line lies in the tangent space of the variety iff the line meets the variety with multiplicity at least 2. Here, we show that through a singular point of an affine variety we can always draw a smooth plane curve inside the ambient affine space, which has the multiplicity of intersection with the variety at least 3. This basic result does not seem to appear anywhere in the literature.

Let K be an arbitrary field. An algebraic curve $C \subset \mathbb{A}^n$ can be parameterized at its smooth point p by Taylor’s series

$$G(t) = p + t \cdot v_1 + t^2 \cdot v_2 + \cdots,$$

where t is a local parameter. Using this parameterization, we can define multiplicity of intersection of an affine variety with the curve C at the point p .

Definition 1.1. Let m be a nonnegative integer and $X \subset \mathbb{A}^n$ an affine variety, given by $\mathbf{I}(X) \subset K[x_1, \dots, x_n]$. Suppose that we have a curve C , smooth at a point p and with the parameterization as above. Then C meets X with multiplicity m at p if $t = 0$ is a zero of multiplicity at least m of the series $f \circ G(t)$ for all $f \in \mathbf{I}(X)$, and for some f , of multiplicity exactly m . We denote this multiplicity of intersection by $I(p, C, X)$.

Remark 1.2. This definition does not depend on the choice of the parameterization. To check this one can use the following criterion: $t = 0$ is a zero of multiplicity m of $h(t) \in K[[t]]$ if and only if $h(0) = h'(0) = \cdots = h^{(m-1)}(0) = 0$ but $h^{(m)}(0) \neq 0$.

Also, we denote by $T_p X$ ($TC_p X$) the tangent space (cone) of a variety X at a point p . The next result contains a necessary condition for a line to be in the tangent cone. Unfortunately, it is not a sufficient one.

Theorem 1.3. Let $X \subset \mathbb{A}^n$ be an affine variety and $p \in X$ a singular point. Then for each $v \in TC_p X$ there exists a smooth curve $C \subset \mathbb{A}^n$ through p such that $v \in T_p C$ and $I(p, C, X) \geq 3$.

Proof. Let $v \in TC_p X$. We can choose affine coordinates x_1, \dots, x_n , so that $p = (0, \dots, 0)$. Let C be a curve through p parameterized by $G(t) = t \cdot v + t^2 \cdot \gamma$, where

$v \in TC_p X$, $\gamma = (\gamma_1, \dots, \gamma_n)$. It suffices to show that there is γ , which is zero or not a scalar multiple of v , such that $t = 0$ is a zero of multiplicity ≥ 3 of $f \circ G(t)$ for all $f \in \mathbf{I}(X)$. In this case, C will be a smooth curve with the local parameter t at p .

For each $f \in \mathbf{I}(X)$, denote by $l_f(x) = l_1 x_1 + \dots + l_n x_n$ and $q_f(x)$ the linear and quadratic parts of f . According to this notation we define

$$W = \{(l_1, \dots, l_n, q_f(v)) : f \in \mathbf{I}(X)\} \subset k^{n+1},$$

which is a subspace since $\mathbf{I}(X)$ is. For all $f \in \mathbf{I}(X)$, we have

$$f(t \cdot v + t^2 \cdot \gamma) = l_f(t \cdot v) + l_f(t^2 \cdot \gamma) + q_f(t \cdot v) + \text{terms of degree } \geq 3 \text{ in } t.$$

Since $v \in T_p X$, we get $l_f(v) = 0$ for any $f \in \mathbf{I}(X)$. Hence,

$$f(t \cdot v + t^2 \cdot \gamma) \equiv t^2(l_1 \gamma_1 + \dots + l_n \gamma_n + q_f(v))$$

modulo terms of degree ≥ 3 in t . So we need to resolve equations $a_1 \gamma_1 + \dots + a_n \gamma_n + a_{n+1} = 0$ in variables $\gamma_1, \dots, \gamma_n$ for all possible $(a_1, \dots, a_{n+1}) \in W$.

Consider the scalar product $k^{n+1} \times k^{n+1} \rightarrow k$, which sends $a \times b$ to $a \cdot b := \sum_{i=1}^{n+1} a_i b_i$. Using this scalar product, we will find the vector $(\gamma_1, \dots, \gamma_n, 1) \in W^\perp := \{b \in k^{n+1} : b \cdot a = 0 \text{ for all } a \in W\}$. If $W \subseteq k^n \times \{0\}$, then we can put $\gamma = (0, \dots, 0)$ to obtain the required curve. Otherwise, consider the projection $\pi : k^{n+1} \rightarrow k^n \times \{0\}$, which assigns $(a_1, \dots, a_n, 0)$ to (a_1, \dots, a_{n+1}) . And, let $\tilde{\pi} : W \rightarrow k^n \times \{0\}$ be induced by π . The projection $\tilde{\pi}$ is injective, because $v \in TC_p X$ and $l_f = 0$ imply $q_f(v) = 0$. If we denote $\tilde{W} = \tilde{\pi}(W)$, then $\dim W^\perp = \dim \tilde{W}^\perp$, by injectivity of $\tilde{\pi}$. Now, if W^\perp is of the form $W_1 \times \{0\}$, then by construction $\tilde{W}^\perp = W_1 \times k$, contradicting with the equality of the dimensions. Therefore, there exists a vector $(\gamma_1, \dots, \gamma_n, 1) \in W^\perp$, i.e., $l_f(\gamma) + q_f(v) = 0$ for all $f \in \mathbf{I}(X)$. We claim that this γ is linearly independent of v . Indeed, since W is not included into $k^n \times \{0\}$, we get $q_f(v) \neq 0$ for some $f \in \mathbf{I}(X)$, which implies $l_f(\gamma) \neq 0$. On the other hand, since $v \in T_p X$, we get $l_f(v) = 0$. Thus, we have found the desired γ . \square

Remark 1.4. This result cannot be improved in the following sense. For the curve X , given by the equation $x^2 = y^3$, in the affine plane, and $p = (0, 0)$ there is no smooth curve C such that $I(p, C, X) \geq 4$.

Remark 1.5. In the case of analytic varieties the theorem is also valid.

2. An application to the affine schemes of algebras

This section shows that at least theoretically one may still be able to answer the problem of Shafarevich discussed in the introduction by solving quadratic equations arising from the second order obstructions to deformations of algebras. In particular, we find that the vectors, contributing to the discrepancy between the tangent spaces to the scheme of associative multiplications and the scheme of the degree 3 nilpotent multiplications,

can be easily “killed” by the obstructions in many cases. Then Theorem 1.3 implies that the tangent cones of the reduced schemes are the same, which means that the irreducible component of one variety is the component of the other one.

Let us recall the notation from [1,3]. The affine scheme C_n of all multiplications on a fixed n -dimensional vector space V over a field K ($\text{char } K \neq 2$), which represent associative commutative algebras, is given by the equations of commutativity and associativity:

$$c_{ij}^k = c_{ji}^k, \quad \sum_{s=1}^n c_{ij}^s c_{sk}^l = \sum_{s=1}^n c_{is}^l c_{jk}^s$$

in the structure constants of multiplication $e_i e_j = \sum_{k=1}^n c_{ij}^k e_k$ of the basis $\{e_1, \dots, e_n\}$. Similar equations determine the affine scheme A_n of commutative nilpotent degree 3 multiplications. The reduced schemes associated to the above schemes are denoted C_n^{red} and A_n^{red} , respectively. From [3], the irreducible components of A_n^{red} have a very simple description

$$A_{n,r} = \{N \in A_n^{\text{red}} \mid \dim N^2 \leq r \leq \dim \text{Ann}_N N\},$$

$1 \leq r \leq (n-1)(n-r+1)/2$, where N denotes the algebra represented by the corresponding multiplication, N^2 is the square of the algebra and Ann is the annihilator.

Let $d := n - r$, then, for $r = 1, 2$ and $r > (d^2 - 1)/3$, Shafarevich in [3] showed that $A_{n,r}$ is not a component of C_n^{red} , by constructing a line, contained in C_n^{red} but not in A_n^{red} , through a point of $A_{n,r}$. For other r , one has to compare the tangent spaces to the non-reduced schemes A_n and C_n . The smooth set of A_n^{red} is the union of

$$U_{n,r} = \{N \in A_n^{\text{red}} \mid \dim N^2 = r = \dim \text{Ann}_N N\}.$$

If $W \subset V$ is a subspace of dimension r , then the space $S_{n,r} = L(S^2(V/W), W)$ of all linear maps from the symmetric product of V/W to W is naturally included as an affine subspace into $A_{n,r}$. The group $G = \text{GL}(V)$ acts on C_n and $GS_{n,r} = A_{n,r}$. Then, the tangent space to $A_{n,r}$ at a point $N \in S_{n,r}$ is

$$T_N A_{n,r} = L(S^2(N/N^2), N^2) + T_N G N,$$

where the tangent space $T_N G N$ to the orbit is the space of maps from $L(S^2 N, N)$ given by the coboundaries (in a Hochschild complex, see [1]) $x \circ y = x\varphi(y) - \varphi(xy) + y\varphi(x)$ for some linear map $\varphi: N \rightarrow N$.

The tangent space $T_N C_n$ always includes $T_N A_{n,r} + F$, where the space F consists of $x \circ y = xf(y) + yf(x)$ for some $f \in L(N/N^2, K) \subset L(N, K)$. To explain this difference Shafarevich embedded the scheme C_n into the scheme \tilde{C}_n of commutative associative multiplications on a $(n+1)$ -dimensional space which represent algebras with a unit e :

$$C_n \hookrightarrow \tilde{C}_n, \quad N \mapsto N \oplus Ke.$$

In this situation, the subgroup \tilde{G} of $\mathrm{GL}(V \oplus Ke)$ which fixes the unit acts on \tilde{C}_n . It turns out that $T_N GN + F$ is the tangent space to $\tilde{G}S_{n,r}$, whence the equality

$$T_N \tilde{C}_n = T_N A_{n,r} + F \quad (1)$$

was enough to conclude that $A_{n,r}$ is the component of C_n^{red} . The equality was shown in [3] for $3 \leq r \leq (d+1)(d+2)/6$, and this also holds $r = (5d^2 - 8d)/16$ and d divisible by 4 by [1, Section 2.1].

The original Shafarevich's method is very special to this situation and is impractical to use it in general to explain the nilpotents in the structure sheaf of a scheme. We expect that the equality (1) holds for all $3 \leq r < (d^2 - 1)/3$, which would leave only one case $r = (d^2 - 1)/3$ unsettled. However, in this last case the tangent space to C_n at a generic point in $A_{n,r}$ is too big:

$$T_N C_n = L(S^2(N/N^2), N^2) + T_N GN + L(S^2 N_1, N_1), \quad (2)$$

where $N = N_1 \oplus N^2$ is a fixed decomposition. To show this one have to use Lemma 1 in [1]: a generic algebra $N \in A_{n,r}$, for $r \leq (d^2 - 1)/3$, can be given by d generators and a $(d(d+1)/2 - r)$ -dimensional space of homogeneous degree 2 relations among the generators, so that the nilpotence of the third degree follows from this relations. But, for $r = (d^2 - 1)/3$, this condition implies that there are no nontrivial relations among the homogeneous degree 2 relations, i.e., all of the relations $\sum_i n_i z_i = 0$ in $S^3 N_1$, $n_i \in N_1$, $z_i \in Z := \ker(\mu : S^2 N_1 \rightarrow N^2)$ (μ is the multiplication on N), are induced by a trivial in $N_1 \otimes Z$ element $\sum_i n_i \otimes z_i$. Indeed, $\dim S^3 N_1 = d(d+1)(d+2)/6$, while the number of equations that we get from the homogeneous degree 2 relations is equal to

$$\dim N_1 \otimes Z = d(d(d+1)/2 - (d^2 - 1)/3) = d(d+1)(d+2)/6.$$

So, if there is a nontrivial relation among the homogeneous degree 2 relations, there would be not enough equations to deduce $N^3 = 0$. Since the only restriction in [1, Lemma 1] for a vector from $L(S^2 N_1, N_1) \subset L(S^2 N, N)$ to be tangent to the scheme C_n was arising from the nontrivial relations, we conclude (2).

Anan' in suggested in [1] to use the second order obstructions to deformations of algebras to eliminate the excessive space $L(S^2 N_1, N_1)$. To be precise, suppose that $N \in A_{n,r}$ is a smooth point of C_n^{red} and $\circ \in L(S^2 N_1, N_1)$. Then, if \circ is tangent to C_n^{red} , the sum $\circ + L(S^2(N/N^2), N^2)$ also lies in the tangent space $T_N C_n^{\mathrm{red}}$. A deformation of a commutative algebra on a vector space V with multiplication $x \cdot y$ can be thought as a smooth curve

$$x \cdot y + (x \circ y)t + (x \star y)t + \dots$$

in the affine space $L(S^2 V, V)$, where t is a local parameter. It is a well know fact in algebraic geometry that through a smooth point p of a variety X we can find a smooth curve inside X , whose tangent vector at p is a given one in $T_p X$. Therefore, the smooth

curves in C_n^{red} with tangent vectors $\circ + L(S^2(N/N^2), N^2)$ give a lot of equations (the second order obstructions) arising from the associativity of multiplication:

$$(x \circ y) \circ z - x \circ (y \circ z) = x(y \star z) - (xy) \star z + x \star (yz) - (x \star y)z$$

for some $\star \in L(S^2N, N)$, where $\tilde{\circ} \in \circ + L(S^2(N/N^2), N^2)$. A nice thing about these quadratic equations on \circ is that one can linearize them:

$$\begin{aligned} (x \circ y) \star z + (x \star y) \circ z - x \circ (y \star z) - x \star (y \circ z) \\ = x(y \star z) - (xy) \star z + x \star (yz) - (x \star y)z \end{aligned}$$

for all $\star \in L(S^2(N/N^2), N^2)$. These equations have been used to prove Theorem 1 in [1] which implies that $A_{n,r}$ is the component of C_n^{red} for $d(d+1)/9 \leq r \leq [d/3](d-3)$ (almost all cases that are not covered by [3]) if a certain algebra $N \in A_{n,r}$ is a smooth point of C_n^{red} . Unfortunately, it seems impossible to prove that N is the smooth point unless we know the tangent space at N .

Now, we can apply Section 1. Let $N \in A_{n,r}$ with multiplication $x \cdot y$ and $\circ \in TC_N C_n^{\text{red}}$. By Theorem 1.3, there exists a smooth plane curve C

$$x \cdot y + (x \circ y)t + (x \star y)t^2$$

through N in the affine space $L(S^2V, V)$ of commutative multiplications on the vector space V , such that \circ is the tangent vector to the curve at N and the multiplicity of intersection $I(p, C, C_n^{\text{red}}) \geq 3$. Hence, the associativity equations in the structure constants imply

$$(x \circ y) \circ z - x \circ (y \circ z) = x(y \star z) - (xy) \star z + x \star (yz) - (x \star y)z \quad (3)$$

for some $\star \in L(S^2N, N)$. Note that we do not assume the smoothness condition at N .

We want to deduce the effective part of the obstructions to \circ . As in [1, §1], decompose \circ and \star into the sum of linear maps f_{ij}^k and g_{ij}^k in $L(N_i \otimes N_j, N_k)$, respectively, where $1 \leq i, j, k \leq 2$ and $N = N_1 \oplus N_2$, $N_2 := N^2$. For a sufficiently generic $N \in A_{n,r}$, besides commutativity, f_{ij}^k satisfy

$$\begin{aligned} f_{12}^1 = f_{22}^1 = f_{22}^2 = 0, \\ xf_{11}^1(y, z) + f_{12}^2(x, yz) = f_{12}^2(z, xy) + f_{11}^1(x, y)z, \end{aligned} \quad (4)$$

by [1, §1]. Moreover, f_{12}^2 satisfying (4) is unique for a given f_{11}^1 , while its existence condition is described in [1, Lemma 1]. The missing $f_{11}^2 \in L(S^2(N/N^2), N^2)$ is always tangent to C_n .

From (3) we get

$$f_{11}^1(f_{11}^1(x, y), z) - f_{11}^1(x, f_{11}^1(y, z)) = -g_{12}^1(z, xy) + g_{12}^1(x, yz), \quad (5)$$

$$f_{12}^2(x, f_{12}^2(y, zt)) - f_{12}^2(y, f_{12}^2(x, zt)) = xg_{12}^1(y, zt) - yg_{12}^1(x, zt), \quad (6)$$

and

$$f_{12}^2(f_{11}^1(x, y), zt) - f_{12}^2(x, f_{12}^2(y, zt)) = xg_{12}^1(y, zt) - g_{22}^2(xy, zt),$$

where $x, y, z, t \in N_1$. The last equation determines g_{22}^2 , and, one can check that commutativity of g_{22}^2 (surprisingly) follows from (4), (5) and (6). So, the only restrictions for \circ to lie in the tangent cone to C_n are (4), (5) and (6). The same argument as in [1, §1] shows that g_{12}^1 , satisfying (5), is unique for a given f_{11}^1 , and the existence condition is similar to [1, Lemma 1]. We have the following result:

Theorem 2.1. *The variety $A_{n,r}$ is an irreducible component of C_n^{red} when $r \leq (d^2 - 1)/3$ if and only if, for some sufficiently generic $N \in A_{n,r}$, a solution $\circ \in L(S^2N, N)$ to (4), (5) and (6) has its part $f_{11}^1 = 0$ on the kernel of the multiplication $\mu: S^2N_1 \rightarrow N^2$ of the algebra N .*

Proof. If $A_{n,r}$ is an irreducible component of C_n^{red} , then, for some sufficiently generic $N \in A_{n,r}$, the tangent cone to C_n^{red} at N coincides with

$$T_N A_{n,r} = L(S^2(N/N^2), N^2) + T_N GN.$$

But we know that the solution to (4), (5) and (6) gives rise to a vector \circ from the tangent cone. Hence, $f_{11}^1 = 0$ on the kernel of the multiplication $\mu: S^2N_1 \rightarrow N^2$, because the vectors of $T_N GN$ are of the form $x \circ y = x\varphi(y) - \varphi(xy) + y\varphi(x)$ for some linear map $\varphi: N \rightarrow N$, while the vectors from $L(S^2(N/N^2), N^2)$ clearly satisfy the property.

Conversely, suppose that a vector $\circ \in L(S^2N, N)$ belongs to the tangent cone to the scheme C_n at N . So, by the discussion above, it satisfies (4), (5) and (6). Hence, \circ has $f_{11}^1 = 0$ on the kernel of the multiplication of some sufficiently generic $N \in A_{n,r}$. Then f_{11}^1 with $f_{12}^2(x, yz) = -xf_{11}^1(y, z)$ and $g_{12}^1(x, yz) = -f_{11}^1(x, f_{11}^1(y, z))$ are unique solutions to (4), (5) and (6). This shows that \circ is actually from $T_N A_{n,r}$. \square

We will finish this section showing the result of [1, Theorem 2] with application of the above theorem and without the use of Shafarevich's embedding $C_n \hookrightarrow \tilde{C}_n$.

Corollary 2.2. *$A_{n,r}$ is a component of C_n^{red} for the values $r = (5d^2 - 8d)/16$, d is divisible by 4.*

Proof. In [1, Section 2.1], it was already shown that $T_N C_n = T_N A_{n,r} + F$ for a sufficiently generic algebra N . This implies that the part f_{11}^1 of a vector from $T_N C_n$ is determined by $f(x)y + f(y)x$ for some $f \in L(N/N^2, K)$, on the kernel of the multiplication μ on N .

The algebra N had the property that the square of all d generators is zero and for each generator u there was a distinct generator v such that their product uv also vanishes in N . Taking $x = y = u$ and $z = v$ in (5), we have

$$f_{11}^1(2f(u)u, v) - f_{11}^1(u, f(u)v + f(v)u) = 0.$$

Since uv and uu are in the kernel of multiplication μ , we further get

$$2f(u)(f(u)v + f(v)u) - f(u)(f(u)v + f(v)u) - f(v)(2f(u)u) = 0,$$

whence $f(u)^2v - f(u)f(v)u = 0$. Therefore, $f(u) = 0$ for all generators, and $f_{11}^1 = 0$ on the kernel of μ as it was required in Theorem 2.1. \square

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